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NOTE ON EVALUATION OF INDETERMINATE FORMS.

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THE following considerations are sometimes useful in facilitating the evaluation of functions which take an indeterminate form when the independent variable vanishes. This virtually includes all indeterminate forms; since if $f(z)$ takes an indeterminate form when $z = a$, we may put $z - a = x$, and if it takes an indeterminate form when $z = \infty$, we may put $1 \div z = x$, in either case we have a function of x which takes a corresponding form when $x = 0$.

In the first place it is evident that we may decompose the function into parts and evaluate them separately, observing only that, if more than one of these parts turn out to be infinite in value, they must be recombined, since we are liable to fall upon the indeterminate form $\infty - \infty$.

Again, we can decompose the function into factors, and evaluate them separately, observing only that, if any factor turns out to be either zero or infinite, it must be treated in connection with the other zero and infinite factors. Now transferring the infinite factors to the denominator we have to deal with a fractional expression composed of zero factors in both terms.

Again, we may substitute for any zero factor any other zero factor *which bears to it the ratio of equality when* $x = 0$; for in so doing we merely multiply the function by a factor which takes the form $\frac{0}{0}$ but is known to have the value unity when $x = 0$.

We may call these zero factors *equivalents*: thus by reason of the familiar results

$$\left.\frac{\sin x}{x}\right]_0 = \left.\frac{\tan x}{x}\right]_0 = \left.\frac{\log(1+x)}{x}\right]_0 = \left.\frac{\varepsilon^x - 1}{x}\right]_0 = 1,$$

we may say that each of the above zero factors is equivalent to x. From these we may of course infer the equivalents of such factors as $\sin mx]_0$, $\tan x^2]_0$, $\log(1-x)]_0$ &c; again, since $(1+\cos x)(1-\cos x)]_0 = 1-\cos^2 x]_0 = \sin^2 x]_0 = x^2]_0$ we have $1-\cos x]_0 = \frac{1}{2}x^2]_0$.

Now if we substitute for each zero factor its equivalent in the form of a power of x, the value of the function is at once apparent.

The equivalents of the elementary forms will readily occur to the memory, and moreover, when the developments of the constituents of a zero factor are known, the equivalent which is simply the lowest term in the resulting development is readily inferred, thus the equivalent of the factor $x - \sin x]_0$ is $\frac{1}{6}x^3]_0$.

To find the equivalent of a zero factor $f(x)]_0$ we may employ the usual method, thus, if $f^n(x)$ is the first of the series of derivatives which does not vanish when $x=0$, then

$$\frac{f(x)}{x^n}\Big]_0=\frac{f^n(0)}{n!},$$

hence the equivalent of $f(x)]_0$ is $f^n(0)\,\frac{x^n}{n!}$, which is the first term of the development of $f(x)$ by Maclaurin's Theorem.

In finding the equivalent of a polynomial factor we may avail ourselves of the equivalents of its elements, the equivalent of the whole being the lowest term resulting from the substitution; remembering however that, if the terms containing the lowest power of x have a vanishing coefficient these terms must still be considered in determining the equivalent, which is thus seen to be of a higher order. Thus if the factor is $\log(1+x)-x+x^2]_0$, since the equivalent of $\log(1+x)]_0$ is x the terms of the first degree vanish; it must not now be assumed that the equivalent is x^2, for it will be found that the equivalent of $\log(1+x)-x]_0$ is $-\frac{1}{2}x^2$, hence that of the whole expression is $\frac{1}{2}x^2$:—

As an example, I apply the foregoing principles to the following which occurs in Todhunter's Diff. Calculus:—

$$\frac{(\varepsilon^x-\varepsilon^{-x})^2-2x^2(\varepsilon^x+\varepsilon^{-x})}{x^4}\Big]_0.$$

The equivalent of $\varepsilon^x-\varepsilon^{-x}]_0$ is $2x$, hence neither of the terms of this expression is finite, but after taking the derivative of each term we have

$$\frac{\varepsilon^{2x}-\varepsilon^{-2x}-x^2(\varepsilon^x-\varepsilon^{-x})-2x(\varepsilon^x+\varepsilon^{-x})}{2x^3}\Big]_0,$$

of which the middle term has a finite value. Evaluating this, and repeating the derivative process for the rest, we have

$$\frac{\varepsilon^{2x}+\varepsilon^{-2x}-(\varepsilon^x+\varepsilon^{-x})-x(\varepsilon^x-\varepsilon^{-x})}{3x^2}\Big]_0-1,$$

which again contains a finite portion, hence proceeding as before we have

$$\frac{2\,(\varepsilon^{2x}-\varepsilon^{-2x})-(\varepsilon^x-\varepsilon^{-x})}{6x}\Big]_0-\frac{2}{3}-1$$

or

$$\frac{8x-2x}{6x}\Big]_0-\frac{2}{3}-1=-\frac{2}{3}.$$